

Infinite-Index Fusion and the Failure of Analytic Descent

Relative multiplication, incidence fibers, and maximal nonclosability

J. Scott Little

Independent Researcher

Status. This paper studies the canonical relative multiplication face

$$\mu_{\text{alg}} : N \odot_A N \longrightarrow N, \quad x \odot_A y \longmapsto xy,$$

through its natural Connes-fusion realization. On the canonical algebraic core the face agrees with the dual operator-valued weight. Finite index is the bounded regime; for a Π_1 subfactor, $\|m\|^2 = [N : A]$. If the inclusion admits an infinite E_A -orthonormal family of unitaries, the corresponding Jones translates are mutually orthogonal while all carrying the same face value. Their normalized averages converge to zero in fusion norm, and the multiplication face becomes maximally nonclosable:

$$\text{mul } \overline{\Gamma(m_0)} = L^2(N), \quad \overline{\Gamma(m_0)} = (L^2(N) \boxtimes_A L^2(N)) \oplus L^2(N).$$

For subgroup inclusions $L(H) \subset L(G)$ the mechanism is fiberwise summation along $G \times_H G \rightarrow G$, with every fiber identified with G/H . The dual operator-valued weight remains well defined in both regimes; the obstruction belongs to the attempted realization of the algebraic face as an ordinary closed Hilbert-space operator.

Research note – not peer reviewed. Trust, but verify. Always.

Contents

1	Introduction	1
2	Canonical relative multiplication	1
3	Finite index and bounded contraction	2
4	Infinite orthonormal multiplicity	3
5	Subgroup incidence geometry	4
6	Algebraic contractions and analytic realization	5
7	Descent defect	5
8	What survives	6
9	Outlook	6
A	Theorem and definition ledger.	6
B	Dependency graph	6
C	Discovery log	7
D	Negative-space ledger	8

1 Introduction

The paper begins with the object that already carries the problem:

$$N \odot_A N \longrightarrow N, \quad x \odot_A y \mapsto xy.$$

Algebraically, nothing is wrong. The balancing relation is compatible with multiplication, and higher adjacent contractions inherit associativity from N . The first real question appears only after the relative tensor product is realized analytically by Connes fusion.

The governing sequence is

$$\text{object} \rightarrow \text{analytic question} \rightarrow \text{criterion} \rightarrow \text{witness} \rightarrow \text{singularity} \rightarrow \text{application}.$$

The term *analytic descent* will be introduced only after the failure has been exhibited concretely.

2 Canonical relative multiplication

Let $A \subset (N, \tau)$ be a unital inclusion of finite von Neumann algebras, with trace-preserving conditional expectation $E_A : N \rightarrow A$. Let e_A be the Jones projection and

$$N_1 = \langle N, e_A \rangle$$

the first basic construction, equipped with its canonical faithful normal semifinite trace Tr_1 . The tracial relative tensor product is identified with

$$L^2(N) \boxtimes_A L^2(N) \cong L^2(N_1, \text{Tr}_1),$$

with algebraic core $x \odot_A y \leftrightarrow xe_A y$.

Set

$$C_A := \text{span } Ne_A N.$$

Proposition 2.1 (Canonical relative multiplication face). *The formula*

$$m_0 \left(\sum_{i=1}^k x_i e_A y_i \right) = \sum_{i=1}^k x_i y_i$$

defines a well-defined linear map $m_0 : C_A \rightarrow N$.

Proof. Suppose $z = \sum_i x_i e_A y_i = 0$ and set $s = \sum_i x_i y_i$. For every $a \in N$,

$$0 = \text{Tr}_1(az) = \tau(as).$$

Taking $a = s^*$ gives $\tau(s^*s) = 0$. Faithfulness of τ gives $s = 0$. □

Let $\widehat{E}_A : N_1^+ \rightarrow \widehat{N}_+$ be the canonical dual operator-valued weight. On its finite ideal it has the usual linear bimodular extension.

Proposition 2.2 (Dual-weight realization). *One has $\widehat{E}_A(e_A) = 1$, $C_A \subset \mathfrak{m}_{\widehat{E}_A}$, and*

$$m_0 = \widehat{E}_A|_{C_A}.$$

Proof. For $b \in N_+$,

$$\mathrm{Tr}_1(b^{1/2}e_A b^{1/2}) = \tau(b) = \tau(b^{1/2}\widehat{E}_A(e_A)b^{1/2}),$$

so faithfulness gives $\widehat{E}_A(e_A) = 1$. Bimodularity then yields

$$\widehat{E}_A(xe_A y) = x\widehat{E}_A(e_A)y = xy.$$

□

3 Finite index and bounded contraction

Lemma 3.1 (Bounded descent forces a Pimsner–Popa inequality). *If m_0 extends boundedly with $\|m\| \leq C$, then*

$$E_A(x) \geq C^{-2}x, \quad x \in N_+.$$

Proof. For $x, y \in N$,

$$\|xy\|_2^2 \leq C^2 \|xe_A y\|_{2, \mathrm{Tr}_1}^2 = C^2 \tau(y^* E_A(x^* x) y).$$

Hence

$$\tau(y^* [C^2 E_A(x^* x) - x^* x] y) \geq 0$$

for every $y \in N$. A negative spectral projection would contradict faithfulness of τ , so $x^* x \leq C^2 E_A(x^* x)$. □

We use the standard finite-index equivalence between a positive Pimsner–Popa bound, finite completely-positive index, and boundedness of the dual operator-valued weight. The optimal constants need not coincide outside the factor case.

Proposition 3.2 (Norm of the bounded face). *Assume $\widehat{E}_A$ is bounded and put*

$$h = \widehat{E}_A(1) \in Z(N)_+.$$

Then the induced map

$$m : L^2(N_1, \mathrm{Tr}_1) \rightarrow L^2(N, \tau)$$

satisfies

$$\|m\|^2 = \|h\|.$$

Proof. Normalize by h on its support and apply Kadison–Schwarz to obtain

$$\widehat{E}_A(z)^* \widehat{E}_A(z) \leq h \widehat{E}_A(z^* z).$$

Taking τ gives $\|m\|^2 \leq \|h\|$. For the reverse inequality, let

$$p_\varepsilon = 1_{(\|h\| - \varepsilon, \|h\|]}(h).$$

Then $p_\varepsilon \in L^2(N_1, \text{Tr}_1)$ and

$$\frac{\|m(p_\varepsilon)\|_2^2}{\|p_\varepsilon\|_{2, \text{Tr}_1}^2} = \frac{\tau(p_\varepsilon h^2)}{\tau(p_\varepsilon h)} \geq \|h\| - \varepsilon.$$

Let $\varepsilon \downarrow 0$. □

Theorem 3.3 (Finite-index bounded regime). *The canonical face m_0 extends boundedly if and only if the trace-preserving expectation satisfies the equivalent finite-index conditions recalled above. When this holds,*

$$\|m\|^2 = \|\widehat{E}_A(1)\|.$$

If $A \subset N$ is an inclusion of II_1 factors, then

$$\|m\|^2 = [N : A].$$

4 Infinite orthonormal multiplicity

Definition 4.1 (Maximal nonclosability). Let $T : D(T) \subset X \rightarrow Y$ be densely defined. Set

$$\text{mul } \overline{\Gamma(T)} := \{y \in Y : (0, y) \in \overline{\Gamma(T)}\}.$$

We call T *maximally nonclosable* if

$$\text{mul } \overline{\Gamma(T)} = Y.$$

Lemma 4.2 (Total graph closure). *If $D(T)$ is dense and $\text{mul } \overline{\Gamma(T)} = Y$, then*

$$\overline{\Gamma(T)} = X \oplus Y.$$

Proof. The hypothesis gives $\{0\} \oplus Y \subset \overline{\Gamma(T)}$. For $x \in D(T)$, subtract $(0, Tx)$ from (x, Tx) to obtain $(x, 0)$ in the closed relation. Density gives $X \oplus \{0\}$, and hence the entire product. □

Definition 4.3 (E_A -orthonormal unitary family). A family $\{u_i\}_{i \in I} \subset \mathcal{U}(N)$ is E_A -orthonormal if

$$E_A(u_i^* u_j) = \delta_{ij} 1.$$

Lemma 4.4 (Orthogonal Jones witnesses). *If $\{u_i\}$ is E_A -orthonormal and*

$$p_i = u_i e_A u_i^*,$$

then

$$p_i p_j = \delta_{ij} p_i, \quad \text{Tr}_1(p_i) = 1, \quad m_0(p_i) = 1.$$

Proof. Use $e_A x e_A = E_A(x) e_A$, the unitary relation, and the defining trace normalization. □

Theorem 4.5 (Maximal fusion-face singularity). *Suppose there exists an infinite E_A -orthonormal family of unitaries $\{u_i\}_{i \geq 1} \subset \mathcal{U}(N)$. Then*

$$\text{mul } \overline{\Gamma(m_0)} = L^2(N, \tau),$$

and consequently

$$\overline{\Gamma(m_0)} = L^2(N_1, \text{Tr}_1) \oplus L^2(N, \tau).$$

Proof. Set $p_i = u_i e_A u_i^*$ and

$$\xi_n = \frac{1}{n} \sum_{i=1}^n p_i.$$

Orthogonality gives

$$\|\xi_n\|_{2, \text{Tr}_1}^2 = \frac{1}{n} \rightarrow 0,$$

while

$$m_0(\xi_n) = 1.$$

For every $z \in N$,

$$z \xi_n \rightarrow 0, \quad m_0(z \xi_n) = z.$$

Thus $N \subset \overline{\text{mul } \Gamma(m_0)}$. The multivalued part of a closed linear relation is closed, and N is dense in $L^2(N, \tau)$, so the multivalued part is all of $L^2(N, \tau)$. Apply the preceding lemma. $\square$

Remark 4.6 (Scope). Infinite index alone is not used in the preceding theorem. The hypothesis is infinite E_A -orthonormal unitary multiplicity. The subgroup case below provides a canonical source of such families.

5 Subgroup incidence geometry

Let $H \leq G$ be a subgroup of a countable discrete group and set $A = L(H) \subset L(G) = N$. Distinct right cosets give an E_A -orthonormal family $u_i = \lambda_{g_i}$.

Corollary 5.1 (Infinite-index subgroup inclusions). *If $[G : H] = \infty$, then the canonical relative multiplication face is maximally nonclosable.*

Define

$$G \times_H G = (G \times G) / \sim, \quad (gh, k) \sim (g, hk).$$

Proposition 5.2 (Incidence realization). *There is a canonical unitary*

$$L^2(L(G)) \boxtimes_{L(H)} L^2(L(G)) \cong \ell^2(G \times_H G)$$

under which the multiplication face becomes the pushforward along

$$\mu : G \times_H G \rightarrow G, \quad \mu([g, k]) = gk.$$

Proposition 5.3 (Incidence fibers). *For every $r \in G$,*

$$\mu^{-1}(r) \cong G/H.$$

Proof. The map

$$gH \mapsto [g, g^{-1}r]$$

is well defined and bijective. $\square$

Theorem 5.4 (Subgroup index dichotomy). *If $[G : H] = d < \infty$, then the pushforward extends boundedly and*

$$\|\mu_*\| = \sqrt{d}.$$

If $[G : H] = \infty$, then its restriction to the finitely supported algebraic core is maximally nonclosable.

Proof. Finite fibers are controlled by Cauchy–Schwarz. Equality is attained by a normalized constant vector on a single fiber. For an infinite fiber, choose distinct $s_i \in \mu^{-1}(r)$ and set

$$f_n = \frac{1}{n} \sum_{i=1}^n \delta_{s_i}.$$

Then $\|f_n\|_2 \rightarrow 0$ while $\mu_* f_n = \delta_r$. The maximal conclusion follows from the preceding subgroup corollary. $\square$

6 Algebraic contractions and analytic realization

For $r \geq 1$ define

$$B_{\text{alg}}^{(r)} = N^{\odot_{A^r}}.$$

Adjacent multiplication maps

$$c_i^{(r)}(x_1 \odot \cdots \odot x_r) = x_1 \odot \cdots \odot (x_i x_{i+1}) \odot \cdots \odot x_r$$

satisfy the usual associativity identities. The natural analytic carriers are

$$X^{(r)} = L^2(N)^{\mathbb{B}_{A^r}}.$$

Theorem 6.1 (Failure of face-compatible completion). *For an infinite-index subgroup inclusion, the canonical adjacent multiplication contractions on the algebraic relative tensor powers cannot all extend as bounded or closable Hilbert-space contractions on the corresponding levelwise Connes-fusion completions. The obstruction occurs already at depth two.*

7 Descent defect

The preceding theorem identifies a failure before any additional categorical structure is introduced. Only now do we name it.

Definition 7.1 (Descent defect). Let $j : D \rightarrow X$ have dense image and let $J : D \rightarrow Y$. Define

$$\text{Def}(J; j) = \{y \in Y : \exists d_n \in D, j(d_n) \rightarrow 0, J(d_n) \rightarrow y\}.$$

Proposition 7.2 (Descent defect and graph closure). *If J induces a densely defined Hilbert-space operator T_0 on $j(D)$, then*

$$\text{Def}(J; j) = \text{mul } \overline{\Gamma(T_0)}.$$

Hence T_0 is closable if and only if $\text{Def}(J; j) = \{0\}$.

For an infinite-index subgroup inclusion,

$$\text{Def}(\mu_{\text{alg}}; j_2) = L^2(L(G)).$$

This is the failure of analytic descent studied here.

8 What survives

The dual operator-valued weight $\widehat{E}_A$ remains normal, faithful, and semifinite even when the L^2 -face is maximally nonclosable. The conclusion is therefore not that composition fails. Rather, the natural Hilbert completion does not support this algebraic face as an ordinary closed operator.

9 Outlook

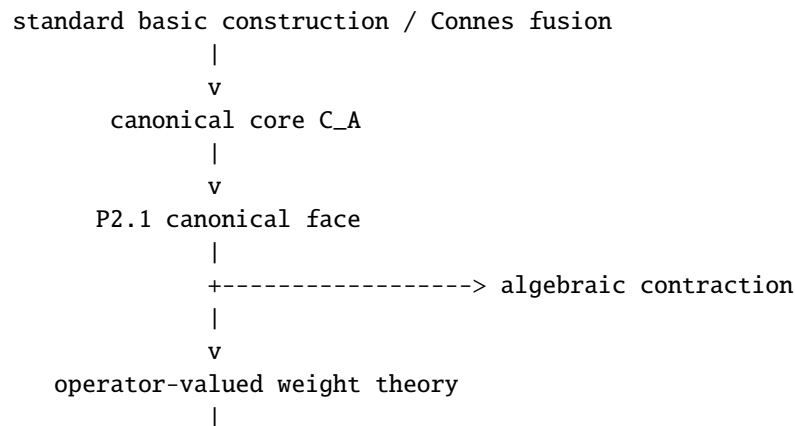
The present paper stops at the first obstruction. It does not construct a replacement reconstruction category, presheaf, stack, or higher invariant. Its conclusion is narrower:

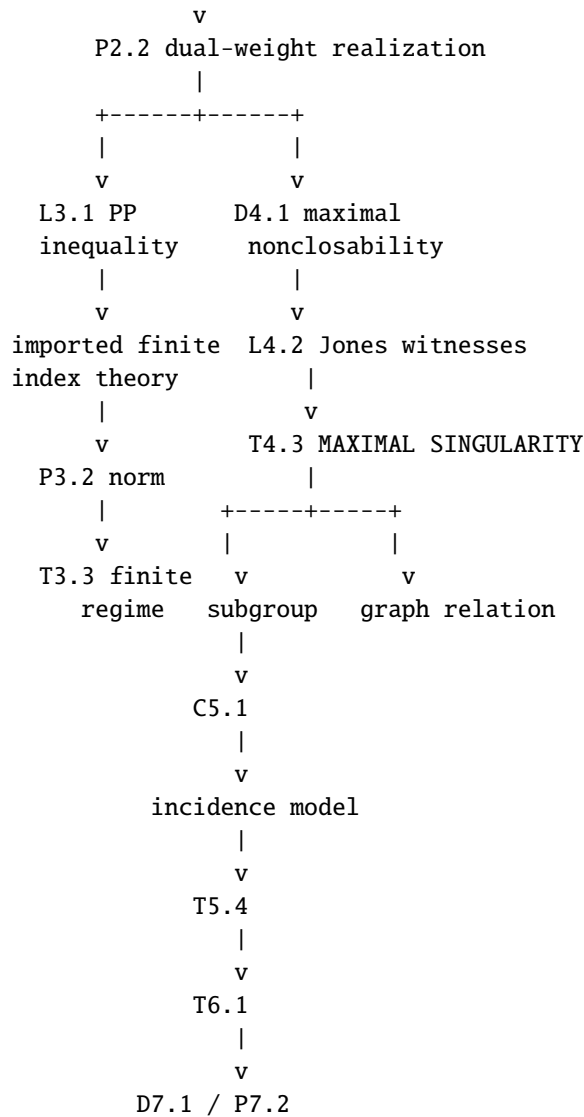
algebraic composability $\Rightarrow$ Hilbert-space descendability.

A Theorem and definition ledger

ID	Item	Status
P2.1	Canonical face m_0	proved
P2.2	Dual-weight realization	proved + imported infrastructure
L3.1	Bounded face $\Rightarrow$ PP inequality	proved
I3	Finite-index equivalences	imported
P3.2	Exact L^2 norm	proved
T3.3	Finite-index bounded regime	certified
D4.1	Maximal nonclosability	definition
L4.2	Orthogonal Jones witnesses	proved
T4.3	Maximal fusion-face singularity	certified
C5.1	Infinite subgroup corollary	certified
P5.2	Incidence realization	proved / standard geometry
P5.3	Fiber identification	proved
T5.4	Subgroup index dichotomy	certified
T6.1	Failure of face-compatible completion	consequence
D7.1	Descent defect	definition
P7.2	Descent defect / graph relation	elementary

B Dependency graph





C Discovery log

1. **Original reconstruction problem.** A bare factor appeared unable to retain a group-theoretic datum. The first move was to look for a better object carrying the comparison.
2. **Diagonal inclusion.** The inclusion $L(\text{diag } \Gamma) \subset L(\Gamma \times \Gamma)$ became the natural carrier of additional structure.
3. **Relative multiplication.** The basic construction forced the question whether $N \odot_A N \rightarrow N$ survives Connes-fusion completion.
4. **Coset witness.** Subgroup cosets produced orthogonal Jones translates p_i with the same face value $m_0(p_i) = 1$.
5. **Nonclosability.** Normalized averages converged to zero while their multiplication value stayed fixed.
6. **Maximality.** Left amplification and L^2 -density promoted one nonzero defect value to the entire target.

7. **Incidence geometry.** The subgroup tensor product became $G \times_H G$, with multiplication as pushforward along $[g, k] \mapsto gk$ and fibers G/H .
8. **Generalization beyond groups.** The proof used only infinite E_A -orthonormal unitary multiplicity. The subgroup theorem became an application.
9. **Dual operator-valued weight.** The same face was recognized as $\widehat{E}_A|_{C_A}$, showing that the operation survives outside the ordinary L^2 -operator category.
10. **Analytic descent.** Only after the explicit failure was established was the defect abstracted as $\text{Def}(J; j)$.
11. **Scope freeze.** Presheaves, reconstruction fibers, higher coherence, and the original rigidity problem were deliberately excluded from Paper I.

D Negative-space ledger

The manuscript does not claim:

- $[N : A] = \infty$ implies maximal nonclosability for arbitrary inclusions;
- the incidence model is novel;
- Connes fusion itself fails;
- the dual operator-valued weight is pathological;
- operator-valued weights automatically form a completed bar object;
- a presheaf or higher reconstruction theorem;
- any result on Connes rigidity itself.

References

- [1] U. Haagerup, *Operator valued weights in von Neumann algebras I*, J. Funct. Anal. 32 (1979), 175–206.
- [2] U. Haagerup, *Operator valued weights in von Neumann algebras II*, J. Funct. Anal. 33 (1979), 339–361.
- [3] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) 19 (1986), 57–106.
- [4] M. Frank and E. Kirchberg, *On conditional expectations of finite index*, J. Operator Theory 40 (1998), 87–111.
- [5] D. Sherman, *Relative tensor products for modules over von Neumann algebras*, Contemp. Math. 328 (2003), 275–291.
- [6] D. Penneys, *A planar calculus for infinite index subfactors*, Comm. Math. Phys. 319 (2013), 595–648.